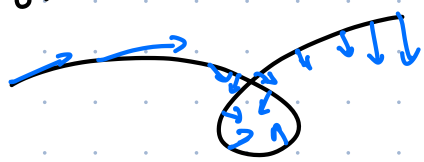
Lecture 31 $\;S_p : X \to X$

One more noncpt example: $SL_n\mathbb{C}/SU(n) = \{\text{pos def herm forms on } \mathbb{C}^n \text{ s.t. } \mathrm{vol}(\Delta^n) = \pi^n\}$

Curvature: A rapid introduction.

(M,g) Riem, $\gamma : I \to M$ smooth. Vec field on γ is map $V : I \to TM$ s.t. $\pi \circ V = \gamma$.



g induces a preferred class of vec fields, the ones that are "g-parallel"

Properties: g-parallel $\Rightarrow$

$V(t_0)$ determines $V(t)$ $\forall t$.

$\forall v_0 \in T_{\gamma(t_0)}M$, $\exists!$ parallel field on γ with $V(t_0) = v_0$

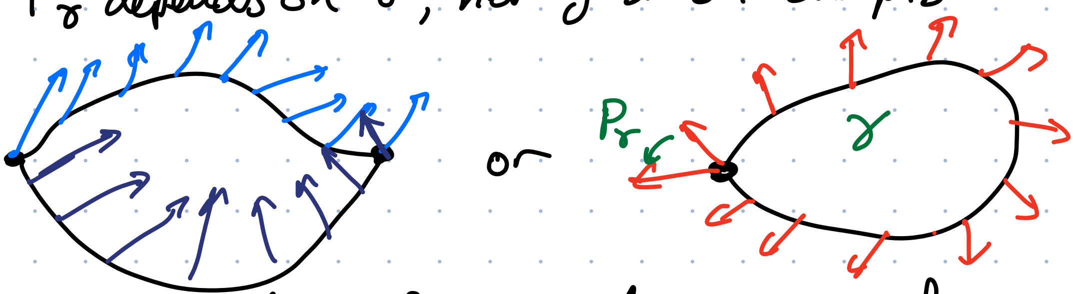
A locally dist-min curve w/ $\|\gamma'(t)\| = 1$ has γ' parallel. $(\Leftarrow)$

For a surface in $\mathbb{R}^3$, parallel means that if you compute V' in $\mathbb{R}^3$ coord that it is orthogonal to the surface.

$P_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M$ parallel transport operator.

$v \longmapsto$ value at 1 of parallel field w/ value v at 0.

Key point: P_γ depends on γ, not just on endpts.

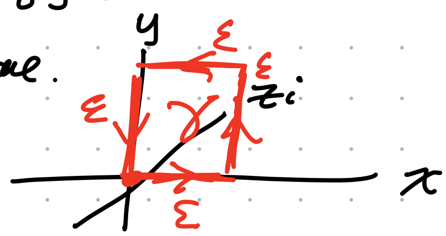


Idea: Curvature is the infinitesimal measure of path dep.

$$R : \underset{\text{at } p}{(\text{tiny loops})} \longrightarrow \mathrm{Aut}(T_pM)$$

__Actual def.__ Instead of a "tiny loop", we take two vectors $v, w \in T_pM$ and make a coord system $(x, y, z_1, \dots, z_{n-2})$ where $p \to 0$, $\frac{\partial}{\partial x}\big|_0 = v$, $\frac{\partial}{\partial y}\big|_0 = w$.

Then we make the square $\gamma_\varepsilon^{v,w}$.



Then define $R_p(v,w) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^2}\left(I - P_{\gamma_\varepsilon^{v,w}}\right) \in \mathrm{Aut}(T_pM)$

i.e. $R_p(v,w)x = \lim \frac{1}{\varepsilon^2}\left(x - P_{\gamma_\varepsilon}(x)\right) \in T_pM$.

__Thm.__ P_γ path indep $\iff R(v,w) = 0 \ \forall v, w$.

R is called the Riemann curvature tensor.

There is also $R(v,w)(x,y) := g(R(v,w)x, y)$. (4-vecs $\to \mathbb{R}$)

For a 2-manifold the entire thing is det by $R(e_1,e_2)(e_1,e_2)$ where e_1, e_2 orthonorm. This scalar is called the Gaussian curvature.

$\mathbb{H}^2$: Distance between two rays from a point grows like $e^{\sqrt{-K}\,d}$

$K<0$

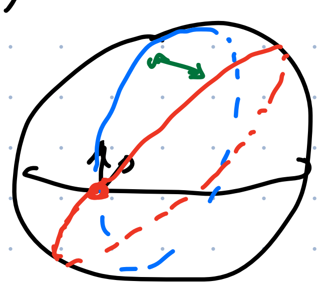
$\left(\frac{1}{y^2}dx^2 + dy^2 \quad K=-1\right)$

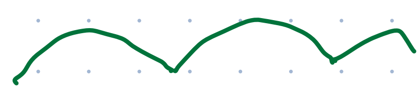
$\mathbb{E}^2$ Dist ... grows linearly

$K=0$

S^2 Dist ... oscillates

$K>0$





More generally a mfld is positively curved if $\underbrace{R(v,w)(v,w)}_{\text{measures behavior in } v,w \text{ plane.}} > 0$ $\forall\, v,w$. Neg curv is similar.

$Exp_p : T_pM \to M$

$Exp_p(v)$ = endpoint of geod from p in direction v of length $\|v\|$.

smooth.

Thm (Cartan-Hadamard) If M is complete, simply conn, and nonpositively curved, then Exp_p is a diffeo.

Also, uniquely geodesic.

Curvature of symm spaces Let $X = G/K$ be a locally symm space.

$R(x,y)$ is G-inv't because the metric is. So at eK suffices.

$T_{eK} G/K \cong \mathfrak{p}$ where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Thm. Let $v,w \in \mathfrak{p}$ regarded as tangent vectors of G/K at the base point. Then $R(v,w) = ad_{\underbrace{[v,w]}_{\text{in } \mathfrak{k}}} : \mathfrak{p} \to \mathfrak{p}$.

That is $R(v,w)x = [[v,w],x]$

or $R(v,w)(x,y) = B([[v,w],x],y)$.

Claim. $B([[v,w],v],w) \leq 0 \;\; \forall\, v,w$.

B is inv't

$$B([[v,w],v],w) = -B([v,[v,w]],w)$$
$$= B([v,w],[v,w])$$
$$\leq 0 \text{ as } [v,w] \text{ in } \mathfrak{k}$$

Totally geodesic: A submanifold $S \subset M$, (M,g) Riem, is called totally geodesic if geodesics in S w/ $g|_{TS}$ are geod in M.

<u>Triple system</u>: $\mathfrak{s} \subset \mathfrak{g}$ s.t. $[[x,y],z] \in \mathfrak{s}$ if $x, y, z \in \mathfrak{s}$

<u>Thm</u>. let $\mathfrak{s}$ be a triple system in $\mathfrak{g}$ and spse $\mathfrak{s} \subset \mathfrak{p}$. Then $\mathrm{Exp}(\mathfrak{s})$ is a totally geodsic submfld.

<u>Ex</u>. $\mathfrak{p}$ itself

or (max ab subalg of $\mathfrak{p}$) $\rightarrow$ Flat. RETURN TO THIS NEXT TIME